Background and Motivation

Historically, the method in which one passes from classical mechanics to quantum mechanics has been as follows. One begins with a classical system, $(\mathbb{R}^{2n}, \omega, H)$. The vector space $\mathbb{R}^{2n}$ describes the state space, it encodes the generalized positions q^i and momenta p_j of the system. The two form, $\omega \in \bigwedge^2 \mathbb{R}^{2n}$ is the canonical symplectic form on $\mathbb{R}^{2n} \cong T^*\mathbb{R}^n$ and $H = \sum \frac{p_j^2}{2m_j} + V(q)$, with $V \in C^\infty(\mathbb{R}^n)$ describes the energy of the system at a point $(p,q) \in \mathbb{R}^{2n}$. One then "quantizes" the system by mapping the Hamiltonian, H, to the densely defined, closable operator $\hat{H} \colon D(\hat{H}) \to \mathcal{H}$ on the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^n)$. To do this, we replace p_j by $\hat{p}_j = -i\hbar\frac{\partial}{\partial q_j}$ and $V(q)$ by $\hat{V}(q) : \psi \mapsto V(q)\psi(q)$. An important property we notice is the relation between the classical Poisson relations of q_i and p_j and the quantum commutation relations. One verifies by a quick calculation that:

$$
\begin{aligned}
[\hat{q}_i, \hat{p}_j] &= i\hbar\delta_{ij} \\
[\hat{q}_i, \hat{q}_j] = [\hat{p}_i, \hat{p}_j] &= 0 \\
\{q_i, p_j\} &= \delta_{ij} \\
\{q_i, q_j\} = \{p_i, p_j\} &= 0
\end{aligned}
$$

The similarities between the two sets of equations became the inspiration for Dirac's quantization condition. If we say that Q is a linear map from $C^\infty(\mathbb{R}^{2n})$ to $\mathrm{Diff}(\mathbb{R}^{2n})$, it fulfills the quantization condition if $[Q(a), Q(b)] = i\hbar Q(\{a,b\})$ for all $a, b \in C^\infty(\mathbb{R}^{2n})$. Here we already run into serious issues codified in the following theorem:

Theorem 1. *(Groenewald-Van Hove) There does not exist a map $Q \colon \mathcal{P}_{\leq 4}(\mathbb{R}^{2n}) \to \mathrm{Diff}(\mathbb{R}^n)$ such that*

$$
\begin{aligned}
Q(1) &= \mathrm{id} \\
Q(q^j) &= q^j \\
Q(p_j) &= -i\hbar\frac{\partial}{\partial q^j} \\
i\hbar Q(\{f,g\}) &= [Q(f), Q(g)]
\end{aligned}
$$

for all $f, g \in \mathcal{P}_{\leq 4}(\mathbb{R}^{2n})$

This means that we cannot create a quantization that adheres to the Dirac quantization condition strictly, this leads to a modified quantization condition:

$$[Q(f), Q(g)] = i\hbar Q(\{f,g\}) + \hbar^2 R(f,g).$$

So we only enforce Dirac's commutation relation up to order $\hbar$ or "asymptotically". The Moyal product was first introduced to give a correct version of commutators in terms of derivatives of the quantized observables. This then inspired Bayen et al. to write down the first definitions of deformation quantiztion in their seminal 1978 paper [1].

Definitions and Examples

In order to formally define what deformation quantization is, we must define a few concepts relating to formal series in $\hbar$.

Definition 1. Let $(\mathscr{A}, \cdot)$ be an associative algebra over the ring R. We define

$$\mathscr{A}[[\hbar]] := \left\{ \sum_{i\in\mathbb{N}} a_l \hbar^l \,\middle|\, a_l \in \mathscr{A} \right\}$$

as the set of formal power series in $\hbar$ with coefficients in $\mathscr{A}$. This set $\mathscr{A}[[\hbar]]$ inherits the R-algebra structure from $\mathscr{A}$ as follows. Given $a, b \in \mathscr{A}[[\hbar]]$, with $a = \sum a_l \hbar^l, b = \sum b_k \hbar^k$,

$$a \cdot b := \sum_{k,l\in\mathbb{N}} a_l b_k \hbar^{l+k}$$

Definition 2. Let $(\mathscr{A}, \cdot)$ be an associative algebra over R. The triple $(\mathscr{A}, \cdot, \{\cdot,\cdot\})$, with $\{\cdot,\cdot\}\colon \mathscr{A} \times A \to A$, is said to be a Poisson algebra if $\{\cdot,\cdot\}$ is a biderivation of $\cdot$ and defines a Lie bracket on $\mathscr{A}$, i.e. for all $a, b, c \in \mathscr{A}, r \in R$

$$\begin{aligned}
\{a, b\} &= -\{b, a\}, \\
\{ra, b\} &= r\{a, b\}, \\
\{ab, c\} &= a\{b, c\} + \{a, c\}b, \\
0 &= \{\{a, b\}, c\} + \{\{c, a\}, b\} + \{\{b, c\}, a\}
\end{aligned}$$

Definition 3. Let $(\mathscr{A}, \cdot, \{\cdot,\cdot\})$ be a Poisson algebra over $\mathbb{C}$. A bilinear map $\star\colon \mathscr{A}[[\hbar]] \times \mathscr{A}[[\hbar]] \to \mathscr{A}[[\hbar]]$ is said to define a star product if the following conditions are satisfied:

$$\begin{aligned}
f \star g &= f \cdot g + o(\hbar) \\
[f, g]_\star := f \star g - g \star f &= i\hbar\{f, g\} + o(\hbar^2) \\
1 \star f &= f \star 1 = f
\end{aligned}$$

We can expand a star product out as

$$f \star g = \sum \hbar^r C_r(f, g)$$

where each C_r is a bilinear map $\mathscr{A} \times \mathscr{A} \to \mathscr{A}$. When $\mathscr{A} = C^\infty(P)$ for a Poisson manifold P, $\star$ is said to be a differential if each C_r is a bidifferential operator, and natural if C_r is of order maximum r in each input.

We will now furnish some of the most basic examples of $\star$-products, coming from our previous setting.

In order to motivate our first star product on $C^\infty(\mathbb{R}^{2n})$, we will begin by mapping $\mathcal{P} := \mathbb{C}[q,p]$ into $\mathrm{Diff}_{poly}(\mathbb{R})$, the space of differential operators from $\mathbb{C}[q]$ to $\mathbb{C}[q]$.

Definition 4. We define the standard ordering representation, $\rho_s \colon \mathcal{P} \to \mathrm{Diff}_{poly}(\mathbb{R}^n)$ by:

$$q^n p^m \mapsto \hat{q}^n \hat{p}^m$$

which one can extend linearly to all of $\mathbb{C}[q,p]$.

It is clear that ρ_s is injective, so we can create our first star product on $\mathbb{C}[q,p]$ by

$$f \star_s g := \rho_s^{-1}(\rho_s(f) \circ \rho_s(g)).$$

To compute this, we start by computing $\rho_s(q^m p^n) \circ \rho_s(q^j p^k)$.

$$\begin{aligned}
\rho_s(q^m p^n) \circ \rho_s(q^j p^k) &= q^m(-i\hbar)^n \frac{\mathrm{d}^n}{\mathrm{d}q^n}\left((q^j)(-i\hbar)^k \frac{\mathrm{d}^k}{\mathrm{d}q^k}\right) \\
&= q^m(-i\hbar)^{n+k} \frac{\mathrm{d}^n}{\mathrm{d}q^n}\left(q^j \frac{\mathrm{d}^k}{\mathrm{d}q^k}\right) \\
&= q^m(-i\hbar)^{n+k} \sum_{r=0}^{n} \binom{n}{r} \frac{j!}{(j-r)!} q^{j-r} \frac{\mathrm{d}^{k+n-r}}{\mathrm{d}q^{k+n-r}} \\
&= \sum (-i\hbar)^r \frac{n!}{(n-r)!r!} \frac{j!}{(j-r)!} q^{j+m-r} (-i\hbar)^{n+k-r} \frac{\mathrm{d}^{k+n-r}}{\mathrm{d}q^{k+n-r}} \\
&= \sum \frac{(-i\hbar)^r}{r!} \frac{j!}{(j-r)!} q^{j+m-r} \frac{n!}{(n-r)!} (-i\hbar)^{n+k-r} \frac{\mathrm{d}^{k+n-r}}{\mathrm{d}q^{k+n-r}}
\end{aligned}$$

Taking $\rho_s^{-1}(q^{j+m-r}(-i\hbar)$ and replacing $q^m p^n$ and $q^j p^k$ by f and g respectively, we arrive at the formula

$$f \star_s g = \sum \frac{(-i\hbar)^r}{r!} \frac{\partial^r f}{\partial p^r} \frac{\partial^r g}{\partial q^r}.$$

This is a terminating, finite series in $\hbar$, but we can extend this to a formula for all of $C^\infty(\mathbb{R}^2)[[\hbar]]$ by the same formula and to $C^\infty(\mathbb{R}^{2n})[[\hbar]]$ by

$$f \star_s g := \sum_{r \in \mathbb{N}} \frac{(-i\hbar)^r}{r!} \sum_{1 \le j \le n} \frac{\partial^r f}{\partial (p_j)^r} \frac{\partial^r f}{\partial (q^j)^r}$$

Definition 5. Let $\mathscr{A}$ be a Poisson algebra over $\mathbb{C}$ and $\star, \star'$ be star products on $\mathscr{A}[[\hbar]]$. $(\mathscr{A}, \star')$ is said to be isomorphic to $(\mathscr{A}, \star)$ if there exists a bijective linear map $N\colon \mathscr{A} \to \mathscr{A}$ with $N(f \star' g) = Nf \star Ng$.

To generate the Weyl-Moyal and Wick products we saw in our lecture, we can apply the transformations: $N_{wm} = \exp\left(-\frac{\hbar}{2i}\sum_{j\leq n} \frac{\partial}{\partial(q^j)}\frac{\partial}{\partial(p^j)}\right)$ and $N_w = \exp\left(\frac{\hbar}{4}\sum_{j\leq n} \frac{\partial^2}{\partial(q^j)^2} + \frac{\partial^2}{\partial(p^j)^2}\right)$. Before providing examples of star products on cotangent bundles, we will give exposition on Hochschild cohomology, and its use in deformation quantization.

Definition 6. Hochschild Cohomology

We can approach deformations in an algebraic manner using the cohomology theory developed by Hochschild.

Let V be a left R module. We define $\mathrm{HC}^k(V) := \mathrm{Hom}(V^k, V)$, the set of k-linear maps $V^k \to V$. Take $\mathrm{HC}^0(V) := V$.

We define the Gerstenhaber bracket on $\mathrm{HC}^\bullet(V) := \bigoplus_{k\in\mathbb{N}} \mathrm{HC}^k(V)$ as a graded Lie bracket $[\cdot,\cdot]_G\colon \overline{\mathrm{HC}}^{\bar{k}}(V) \times \overline{\mathrm{HC}}^{\bar{l}}(V) \to \overline{\mathrm{HC}}^{\bar{k}+\bar{l}}(V)$, with $\overline{\mathrm{HC}}^{k}(V) := \mathrm{HC}^{k-1}(V), \bar{k} := k-1$. The grading is with respect to the shifted degree.

$$\begin{aligned}[][\alpha,\beta](v_1, v_2, \ldots, v_{\bar{k}+\bar{l}+1}) &= \sum_{i\leq k} (-1)^{\bar{i}\bar{k}} \alpha(v_1, v_2, \ldots, v_{i-1}, \beta(v_i, \ldots, v_{i+\bar{l}}), v_{i+l}, \ldots, v_{\bar{k}+\bar{l}+1}) \\ &\quad - (-1)^{\bar{k}\bar{l}} \sum_{j\leq l} (-1)^{\bar{j}\bar{l}} \beta(v_1, v_2, \ldots, v_{j-1}, \alpha(v_j, \ldots, v_{j+\bar{k}}), v_{j+k}, \ldots, v_{\bar{k}+\bar{l}+1})\end{aligned}$$

for $v_1, \ldots v_{\bar{k}+\bar{l}+1} \in V$. For $C \in \mathrm{HC}^2(V)$, we have

$$\begin{aligned}[][C,C]_G(v_1, v_2, v_3) &= -C(C(v_1,v_2),v_3) + C(v_1, C(v_2,v_3)) - C(C(v_1,v_2),v_3) + C(v_1, C(v_2,v_3) \\ &= -2C(C(v_1,v_2)v_3) + 2C(v_1, C(v_2,v_3))\end{aligned}$$

We see that C defines an associative product on V if and only if $[C,C]_G = 0$

Definition 7. Let V be a left R module and $\mu \in HC^2(V)$ an associative product on V. We define the following coboundary operator $\delta^k\colon HC^k(V) \to HC^{k+1}(V)$ by

$$\delta^k_\mu(\alpha)(x_1, \ldots, x_{k+1}) := -[\mu, \alpha]_G.$$

we also define the following cochain complex

$$\cdots \longrightarrow HC^{k-1}(V) \xrightarrow{\delta^{k-1}_\mu} HC^k(V) \xrightarrow{\delta^k_\mu} HC^{k+1}(V) \longrightarrow \cdots$$

This is the Hochschild complex of V. The cohomology of this complex

$$HH_\mu(V) := \bigoplus_{k\in\mathbb{N}} HH_\mu^k(V)$$

is the Hochschild cohomology of V. We of course have $HH_\mu^k(V) := \ker \delta_\mu^k / \mathrm{im}\delta_\mu^{k-1}$.

We will calculate δ_μ^k explicitly. Plugging in to our Gerstenhaber bracket formula we have

$$\begin{aligned}\delta_\mu^k(\alpha(x_1,\dots,x_{k+1})) &= x_1\alpha(x_2,\dots,x_{k+1}) + (-1)^{k+1}\alpha(x_1,\dots,x_k)x_{k+1} \\ &\quad + \sum_{i=1}^{k}(-1)^i\alpha(x_1,\dots,x_i x_{i+1},\dots,x_{k+1})\end{aligned}$$

For $\alpha \in HC^1(V)$ we have $\delta_\mu^1\alpha(x_1,x_2) = x_1\alpha(x_2) - \alpha(x_1x_2) + \alpha(x_1)x_2$. This means that $\alpha \in \ker\delta_\mu^1$ if and only if α is a derivation over μ. We will now give a cohomological description of deformations.

Definition 8. Let $\mathscr{A}$ be an R algebra. A k-th order deformation of $\mathscr{A}$ is a bilinear map $\mu_k := \mu + \nu\mu_{(1)} + \ldots + \nu^k\mu_{(k)} \colon \mathscr{A}\times\mathscr{A} \to \mathscr{A}[[\nu]]$ such that $[\mu_k,\mu_k] = 0 + o(\nu^{k+1})$. where each $\mu_{(i)}$ is a bilinear map $\mathscr{A}\times\mathscr{A}\to\mathscr{A}$.

Let $\mu_{(k+1)}\colon \mathscr{A}\times\mathscr{A}\to\mathscr{A}$ be a bilinear map. By definition $\mu_{k+1} := \mu_k + \nu\mu_{(k+1)}$ defines a k+1th order deformation of μ if and only if $[\mu_{k+1},\mu_{k+1}]_G = o(\nu^{k+2})$. Since $[\mu_k,\mu_k]_G = o(\nu^{k+1})$ we have

$$\begin{aligned}[\mu_{k+1},\mu_{k+1}]_G &= \left[\mu_k + \nu^{k+1}\mu_{(k+1)}, \mu_k + \nu^{k+1}\mu_{(k+1)}\right]_G \\ &= [\mu_k,\mu_k]_G + \nu^{k+1}\left(\left[\mu_k,\mu_{(k+1)}\right]_G + \left[\mu_{(k+1)},\mu_k\right]\right) + o(\nu^{k+2})\end{aligned}$$

If we look at the ν^{k+1} term we have

$$\sum_{\substack{i+j=k+1,\\ k+1>i,j>0}} \left[\mu_{(i)},\mu_{(j)}\right]_G + \left[\mu_k,\mu_{(k+1)}\right]_G + \left[\mu_{(k+1)},\mu_k\right]_G = \sum_{\substack{i+j=k+1,\\ i,j\geq 0}} \left[\mu_{(i)},\mu_{(j)}\right]_G$$

This term must vanish if μ_{k+1} is to be a $k+1$-th order deformation. We can rewrite this condition as

$$\delta_\mu^2(\mu_{(k+1)}) = -\left[\mu_{(k+1)},\mu_{(0)}\right] = \frac{1}{2}\sum_{\substack{i+j=k+1,\\ i,j\geq 1}} \left[\mu_{(i)},\mu_{(j)}\right].$$

This means that the right hand side must be a coboundary. We see that the right hand side is a cocycle as follows:

$$\begin{aligned}
\delta^3_\mu\left(\sum_{\substack{i+j=k+1,\\ i,j\geq 1}} \left[\mu_{(i)},\mu_{(j)}\right]_G\right) &:= \sum_{\substack{i+j=k+1,\\ i,j\geq 1}} \left[\mu,\left[\mu_{(i)},\mu_{(j)}\right]_G\right]_G \\
&= \sum_{\substack{i+j=k+1,\\ i,j\geq 1}} \left(\left[\mu_{(i)},\left[\mu_{(j)},\mu\right]_G\right]_G + \left[\mu_{(j)},\left[\mu,\mu_{(i)}\right]_G\right]_G\right) \\
&= 2\sum_{\substack{i+j=k+1,\\ i,j\geq 1}} \left[\mu_{(i)},\delta^2_\mu(\mu_{(j)})\right]_G \\
&= \sum_{\substack{i+j+l=k+1,\\ i,j,l\geq 1}} \left[\mu_{(i)},\left[\mu_{(j)},\mu_{(l)}\right]_G\right]_G
\end{aligned}$$

This final sum is equal to zero by the graded Jacobi identity. This means that given a k-th order deformation$\sum \mu_{(i)}\nu^i$, this can be extended to a $k+1$th order deformation if and only if the Hochschild cocycle $\sum \left[\mu_{(i)},\mu_{(k+1-i)}\right]_G$ is a Hochschild coboundary.

We can show that for $\mu' = \sum \mu^i \ni^i$ an element of $HC^2(\mathscr{A})[[\nu]]$, the associativity condition $[\mu',\mu']_G = 0$ is equivalent to requiring that for $\gamma = \mu_* - \mu \in \nu HC^2(\mathscr{A})[[\nu]]$ is a solution to the equation

$$\delta^2_\mu(\gamma) - \frac{1}{2}[\gamma,\gamma]_G = 0$$

the Maurer Cartan equation.

Constant Poisson Structure

We now describe the construction of a start product on a constant Poisson structure.

Given a vector space $\mathcal{F}$ and endomorphisms $\mathcal{V}_1,\mathcal{V}_2\colon F\to F$ we can define an endomorphism $\mathcal{V}_1\otimes\mathcal{V}_2$ by

$$\mathcal{V}_1\otimes\mathcal{V}_2(\xi\otimes\eta) := \mathcal{V}_1(\xi)\otimes\mathcal{V}_2(\eta).$$

One recalls that a constant Poisson structure on a vector space V, with basis x^i can be written as a constant bivector field on V,

$$\pi = \sum P_{ij}\frac{\partial}{\partial x^i}\wedge\frac{\partial}{\partial x^j} = \sum P_{ij}\left(\frac{\partial}{\partial x^i}\otimes\frac{\partial}{\partial x^j} - \frac{\partial}{\partial x^j}\otimes\frac{\partial}{\partial x^i}\right)$$

So π defines an endomorphism on $\mathcal{F}\otimes\mathcal{F} = C^\infty(V)\otimes C^\infty(V)$. We'll continue to call this endomorphism π. We can write $\pi(f,g) = \mu(\pi(f\otimes g))$ where $\mu\colon \mathcal{F}\otimes\mathcal{F}\to\mathcal{F}$ is defined by

$f \otimes g \mapsto fg$. Because $\pi(f \otimes g) \in \mathcal{F} \otimes \mathcal{F}$ we can apply π arbitrarily many times. We can then define the following formal series in ν,

$$\exp(\nu\pi/2)(f \otimes g) := \sum_{j \in \mathbb{N}} \frac{\nu^j}{2^j j!} \pi^j (f \otimes g)$$

Proposition 1. *Let (V, π) be a finite dimensional constant Poisson structure. The map defined by*

$$f \star g := \mu(\exp(\nu\pi/2)(f \otimes g))$$

is a star product on $(C^\infty(V), \{\cdot, \cdot\}, \cdot)$.

Deformation Quantization on Symplectic Manifolds

We will use this star product to construct a star product on symplectic manifolds. This construction comes from Boris Fedosov's 1994 paper [2], though my exposition is largely based on [3]. One recalls that for (V, ω) a finite dimensional real vector space, we have the canonical poisson bracket $\{f, g\} := \omega(X_f, X_g)$ making (V, π) a constant Poisson structure. We denote $\hat{\mathscr{F}}(V)$ as the space of formal power series in the variables that give a basis for V^*, i.e. an arbitrary element of $\hat{\mathscr{F}}$, is $a = \sum_{\alpha \in \mathbb{N}^d} a_\alpha x_1^{\alpha_1} x_2^{\alpha_2} \dots x_d^{\alpha_d}$. Denote

$$\hat{\mathscr{F}}(V)^{\geq j} = \{a \in \hat{\mathscr{F}}(V) | a_\alpha = 0 \text{ for} |\alpha| < j\}$$

and for $k \in \mathbb{N}$ we define

$$E^{\geq k}(V) = \sum_{\substack{i,j \in \mathbb{N}, \\ 2i+j \geq k}} \nu^i \hat{\mathscr{F}}(V)^{\geq j}$$

Since the Moyal product on V preserves polynomials, we can restrict the product on $C^\infty(V)[[\nu]]$ to $\hat{\mathscr{F}}(V)[[\nu]]$ and we have the property

$$E^{\geq k}(V) \star E^{\geq l}(V) \subset E^{\leq k+l}(V).$$

To V we associate an an associative algebra

$$A(V) := E^{\geq 0}(V) \otimes \bigwedge V^*$$

with the product $\circledast \colon A(V) \times A(V) \to A(V)$ given by

$$(F \otimes \xi) \circledast (G \otimes \eta) := (F \star G) \otimes (\xi \wedge \eta)$$

for each $F, G \in E^{\geq 0}(V), \xi, \eta \in \bigwedge V^*$. We have a decreasing sequence of vector spaces

$$A^{\geq k}(V) := E^{\geq k}(V) \otimes \bigwedge V^*$$

and $A(V)/A^{\geq k}(V)$ is a finite dimensional vector space for all $k \in \mathbb{N}$. We have a natural linear map $P_0\colon A(V) \to \mathbb{R}$ defined by $F \otimes \xi \mapsto 0$ if $\xi \in \bigoplus_{i\geq 1} \bigwedge^i V^*$ and $F \otimes 1 \mapsto F(0)$ for all $F \in E^{\geq 0}(V)$, $F(0)$ is the constant term of F.

We now switch gears back to our symplectic manifold (M, ω). We know that each tangent space T_pM is a symplectic vector space with constant Poisson structure π_p. We can consider the Weyl-Moyal product $\star_p$ on $\hat{\mathscr{F}}(T_pM)$ and the induced associative product $\circledast_p$ on $A(T_pM)$. We define a section of the projection $\bigcup_{p\in M} A(T_pM) \to M$ is smooth if the projection onto each of the vector bundles $\bigcup_{p\in M} A(T_pM)/A^{\geq k}(T_pM) \to M$ is smooth. For each $k \in \mathbb{N}$, the space of smooth sections which are $A^{\geq k}(T_pM)$ for each $p \in M$ will be denoted by $\mathscr{A}^{\geq k}(M)$, and $\mathscr{A}(M) := \mathscr{A}^{\geq 0}(M)$. Since we have defined a product on each of the fibers, we can define a product of sections. Given $a \in \mathscr{A}^{\geq k}(M), b \in \mathscr{A}^{\geq l}(M)$ we define $a \circledast b$ as the section of $\mathscr{A}^{\geq k+l}(M)$ with $(a \circledast b)(p) = a(p) \circledast_p b(p)$. By combining the maps $P_{0,p} : A(T_pM) \to \mathbb{R}$ gives us a linear map $P_0 : \mathscr{A}(M) \to C^\infty(M)[[\nu]]$.

Theorem 2. *Let M be a symplectic manifold and $(\mathscr{A}(M), \circledast)$ the associative algebra constructed previously. There exists a derivation $D\colon \mathscr{A}(M) \to \mathscr{A}(M)$ satisfying the following: For each $F \in C^\infty(M)[[\nu]]$ there exists a unique $a_F \in \mathscr{A}(M)$ such that $D(a_f) = 0$, $P_0(a_F) = F$.*

We briefly discuss the construction of such a D.

The D takes the form

$$D = \mathrm{d}^\nabla + [\omega + \gamma, \cdot]$$

where d^∇ denotes a choice of symplectic connection, i.e. a connection on TM such that

$$\iota_X \,\mathrm{d}\,(\omega(Y, Z)) = \omega(\nabla_X Y, Z) + \omega(Y, \nabla_X Z)$$

for all $X, Y, Z \in \Gamma(TM)$. We can use this symplectic connection on $\bigcup_{p\in M} \hat{\mathscr{F}}(T_pM) \to M$. This bundle is technically not a vector bundle, since its fibers are infinite dimensional, but we do get a covariant derivative $\mathrm{d}^\nabla : \mathscr{A}(M) \to \mathscr{A}(M)$. We define $[\cdot, \cdot]$ point wise by $[\cdot, \cdot]_p$ using the following formula

$$[a \otimes \xi, b \otimes \eta]_p := (a \star b - b \star a) \otimes (\xi \wedge \eta).$$

Here ω is identified with an element of $\mathscr{A}(M)$ and $\gamma \in \mathscr{A}^{\geq 3}$ is a section of $\bigcup_{p\in M} \hat{\mathscr{F}}(T_pM)[[\nu]] \otimes T_p^*M$ chosen such that $D^2 = 0$.

Corollary 1. *Let* (M, ω) *be a symplectic manifold,* $(\mathscr{A}(M), \circledast)$ *the algebra from before, and* D *the derivation of* $\mathscr{A}(M)$ *from the preceding theorem. Denote* $\star$ *the bilinear map on* $C^\infty(M)[[\nu]]$ *defined on* F, G *by* $F \star G = P_0(a_F \circledast a_G)$ *where* a_F, a_G *are the elements of* $\ker D$ *with* $P_0(a_F) = F, P_0(a_G) = G$. *Then* $\star$ *defines a star product on* (M, π).

References

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